5.1.4. Asymptotes of the General Rational Algebraic Curve

Assume that $f(x, y) = 0$ be the equation of any rational algebraic curve of the n^{th} degree. Suppose that on arranging the equation of the curve into groups of same terms,

$$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \ldots\ldots + a_{n-1}yx^{n-1} + a_nx^n + b_1y^{n-1} + b_2y^{n-2}x + \ldots\ldots + b_{n-1}yx^{n-2} + b_nx^{n-1} + c_2y^{n-2} + \ldots\ldots + \ldots\ldots = 0. \qquad \ldots\ldots(1)$$

This equation can also be written as

$$x^n\phi_n(y/x) + x^{n-1}\phi_{n-1}(y/x) + \ldots\ldots = 0. \qquad \ldots\ldots(2)$$

Where, $\phi_r(y/x)$ is a polynomial in y/x of degree r.

Let $y = mx + c$ be a line that is not parallel to the y-axis. This equation (2) meets at points whose abscissa is given by:

$$x^n\phi_n\left(\frac{mx+c}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{mx+c}{x}\right) + \ldots = 0;$$

$$\text{Or } x^n\phi_n\left(m+\frac{c}{x}\right) + x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right) + \ldots = 0.$$

Expanding each term of the type $\phi_r(m + c/x)$, by Taylor's theorem, we get

$$x^n[\phi_n(m) + (c/x)\phi'_n(m) + (c^2/2x^2)\phi''_n(m) + \ldots\ldots]$$

$$+ x^{n-1}[\phi_{n-1}(m) + (c/x)\phi'_{n-1}(m) + \ldots\ldots] + x^{n-2}[\phi_{n-2}(m) + (c/x)\phi'_{n-2}(m) + \ldots\ldots] + \ldots\ldots = 0.$$

On arranging the terms according to the descending powers of x, and we obatin

$$x^n \phi_n(m) + x^{n-1}[\phi_{n-1}(m) + c\phi'_n(m)] + x^{n-2}[\phi_{n-2}(m) + c\phi'_{n-1}(m) + \frac{1}{2}c^2\phi''_n(m)] + ... = 0. \quad .....(3)$$

If $y = mx + c$ is an asymptote, then the equation must have two infinite roots and consequently

$$\phi_n(m) = 0, \quad .....(4)$$

And $\phi_{n-1}(m) + c\phi'_n(m) = 0,$ .....(5)

As can be seen on dividing equation (3) by x^n. Solving equations (4) and (5) together, we get the values of m and c, and hence the asymptotes are determined by substituting the corresponding values of m and c in the equation $y = mx + c$.